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New multi-parameter Golay 2-complementary sequences and transforms

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Abstract. In this work, we develop a new unified approach to the so-called generalized Golay–Rudin–Shapiro (GRS) 2-complementary multi-parameter sequences. It is based on a new generalized iteration generating construction.

1. Introduction

Binary ± 1-valued *Golay–Rudin–Shapiro* sequences (2-GRSS) associated with the cyclic group $\mathbf{Z}_2^n$ were introduced independently by Golay [1, 2, 3] in 1949-1951, Shapiro [4,5] and Rudin [6] in 1951. M.J.E. Golay [2] introduced the general concept of "complementary pairs" of finite sequences all of whose entries are ± 1. This was motivated by a highly non-trivial applications of infrared spectrometry. Then he gave an explicit construction for binary Golay complementary pairs of length 2^m and later [3] noted that the construction implies the existence of at least $2^m m!/2$ binary Golay sequences of this length. They are known to exist for all lengths $N = 1^\alpha 10^\beta 26^\gamma$, where α, β, γ are integers and $\alpha, \beta, \gamma \geq 0$ (Turyn, [7]), but do not exist for any length N having a prime factor congruent to modulo 4 (Eliahou et al., [8]). In 1951, H. S. Shapiro [4, 5] introduced what became known, after 1963,as the "Rudin-Shapiro" polynomial pairs. Shapiro's work was entirely in pure mathematics. Budisin [9,10,11] using the work of Sivaswamy [12] gave a more general recursive construction for Golay complementary pairs and showed that the set of all binary Golay complementary pairs of length 2^m obtainable from it coincides with those given explicitly by Golay. For a survey of results on binary and non-binary Golay complementary pairs, see Byrnes [13] and Fan, Darnel [14], respectively. In 1999, Davis and Jedwab [15] gave an explicit description of a large class of Golay complementary sequences in terms of certain cosets of the first order Reed–Muller codes.

Discrete *Fourier-Golay–Rudin–Shapiro Transforms* (FGRST) in bases of different Golay–Rudin–Shapiro sequences can be used in many signal processing applications: multiresolution by discrete orthogonal wavelet decomposition, digital audition, digital video broadcasting, communication systems (OFDM, MCDA), radar, and cryptographic systems.

To build the classical FGRST, the following actors are used: 1) the Abelian group $\mathbf{Z}_2^n$, 2) 2-point Fourier transform $\mathcal{F}_2$, and 3) the complex field $\mathbf{C}$; i.e., these transforms are associated with the triple $\left(\mathbf{Z}_2,\mathcal{F}_2,\mathbf{C}\right)$.In this work, we develop a new unified approach to the so-called generalized complex-, $\mathbf{GF}(p)$-, and Clifford-valued complementary sequences. The approach is associated not with the triple $\left(\mathbf{Z}_2,\mathcal{F}_2,\mathbf{C}\right)$, but with $\left(\mathbf{Z}_2,\mathcal{CS}_2(\varphi,\alpha,\gamma),\mathcal{A}lg\right)$ and $\left(\mathbf{Z}_2,\left\{\mathcal{CS}_2^1(\varphi_1,\alpha_1,\gamma_1),...,\mathcal{CS}_n^1(\varphi_n,\alpha_n,\gamma_n)\right\},\mathcal{A}lg\right)$, where $\mathcal{CS}_2^1(\varphi_1,\alpha_1,\gamma_1)$ and $\left\{\mathcal{CS}_2^1(\varphi_1,\alpha_1,\gamma_1),\mathcal{CS}_2^2(\varphi_2,\alpha_2,\gamma_2),...,\mathcal{CS}_n^1(\varphi_n,\alpha_n,\gamma_n)\right\}$ are a single transform or a set of arbitrary unitary (2×2)-transforms of type $\mathcal{CS}_2(\varphi,\alpha,\gamma)=\begin{bmatrix} e^{i\alpha}\cos\varphi & e^{i\gamma}\sin\varphi \\ e^{-i\gamma}\sin\varphi & -e^{-i\alpha}\cos\varphi\end{bmatrix}$, instead of $\mathcal{F}_2=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ if $\mathcal{A}lg=\mathbf{C}$.

The rest of the paper is organized as follows: in Section 2, the object of the study (*Golay–Rudin–Shapiro* binary sequences) is described. In Section 3, the iteration rule to construct the Golay matrix is introduced. In Section 4, the proposed method based on a new generalized iteration construction is explained to construct generalized multi-parameter Golay–Rudin–Shapiro sequences.

2. The object of the study

We begin with the description of the original Golay 2-complementary ± 1-valued sequences.

Definition 1. Let $\mathrm{com}^0(t):=\left(c_0,c_1,...,c_{N-1}\right)$ and $\mathrm{com}^1(t)=\left(s_0,s_1,...,s_{N-1}\right)$, where $c_i,s_i\in\{\pm 1\}$. Both sequences $\mathrm{com}^0(t)$, $\mathrm{com}^1(t)$ are called the (± 1)*-valued complementary* or *Golay complementary pair* over $\{\pm 1\}$, if $COR^0(\tau)+COR^1(\tau)=N\delta(\tau)$, or $\left(\left|\mathrm{COM}^0(z)\right|^2+\left|\mathrm{COM}^1(z)\right|^2\right)_{|z|=1}=N$, where $COR^0(\tau)$, $COR^1(\tau)$ are the periodic correlation functions of $\mathrm{com}^0(t)$, $\mathrm{com}^1(t)$ and $\mathrm{COM}^0(z)=\mathcal{Z}\left\{\mathrm{com}^0(t)\right\}$, $\mathrm{COM}^1(z)=\mathcal{Z}\left\{\mathrm{com}^1(t)\right\}$ are their $\mathcal{Z}-$transforms. Any sequence, which is a member of a Golay complementary pair, is called the Golay sequence and its $\mathcal{Z}-$transform $\mathrm{COM}_k(z)=\mathcal{Z}\left\{\mathrm{com}_k(t)\right\}$ is called the *Golay-Shapiro-Rudin polynomials*. We use two symbols $\boldsymbol{\alpha}_n\in[0,2^{n-1}-1]=\mathbf{Z}_{2^n}$ and $\mathbf{t}_n\in[0,2^{n-1}-1]=\mathbf{Z}_{2^n}$ to enumerate Golay sequences and the discrete time, respectively. For integer $\boldsymbol{\alpha}_n\in[0,2^{n-1}-1]$ and $\mathbf{t}_n\in[0,2^{n-1}-1]$ we shall use binary codes $\vec{\boldsymbol{\alpha}}_n=\left(\alpha_1,\alpha_2,...,\alpha_n\right)$, $\vec{\mathbf{t}}_n=\left(t_1,t_2,...,t_n\right)$, where $\alpha_i,t_i\in\{0,1\}$. Let $\vec{\boldsymbol{\alpha}}_n=\left(\alpha_1,\alpha_2,...,\alpha_n\right)$ and $\vec{\mathbf{t}}_n=\left(t_1,t_2,...,t_n\right)$ be binary codes, then define

$$\boldsymbol{\alpha}_n=\left|\vec{\boldsymbol{\alpha}}_n\right|=\left|\left(\alpha_1,\alpha_2,...,\alpha_n\right)\right|=\sum_{i=1}^{n}\alpha_{n-i+1}2^{i-1},\ \mathbf{t}_n=\left|\vec{\mathbf{t}}_n\right|=\left|\left(t_1,t_2,...,t_n\right)\right|=\sum_{i=1}^{n}t_{n-i+1}2^{n-i}$$

as integers whose binary codes are $\vec{\boldsymbol{\alpha}}_n=\left(\alpha_1,\alpha_2,...,\alpha_n\right)$ and $\vec{\mathbf{t}}_n=\left(t_1,t_2,...,t_n\right)$, where α_n,t_1 are less significant bits (LSB) and α_1,t_n are most significant bits (MSB) of $\vec{\boldsymbol{\alpha}}_n=\left(\alpha_1,\alpha_2,...,\alpha_n\right)$ and $\vec{\mathbf{t}}_n=\left(t_1,t_2,...,t_n\right)$, respectively. Obviously,

$$\begin{array}{llll}
\vec{\boldsymbol{\alpha}}_1=(\alpha_1)\in\mathbf{Z}_2, & \boldsymbol{\alpha}_1=\alpha_1\in\mathbf{Z}_2, & \vec{\mathbf{t}}_1=(t_1)\in\mathbf{Z}_{2^1}, & \mathbf{t}_1=t_1\in\mathbf{Z}_2,\\
\vec{\boldsymbol{\alpha}}_2=\left(\vec{\boldsymbol{\alpha}}_1,\alpha_2\right)\in\mathbf{Z}_2\times\mathbf{Z}_2=\mathbf{Z}_2^2, & \left(\boldsymbol{\alpha}_1,\alpha_2\right)\in\mathbf{Z}_2\times\mathbf{Z}_2, & \vec{\mathbf{t}}_2=(\vec{\mathbf{t}}_1,t_2)\in\mathbf{Z}_2\times\mathbf{Z}_2=\mathbf{Z}_2^2, & (\mathbf{t}_1,t_2)\in\mathbf{Z}_2\times\mathbf{Z}_2,\\
\vec{\boldsymbol{\alpha}}_3=\left(\vec{\boldsymbol{\alpha}}_2,\alpha_3\right)\in\mathbf{Z}_2^2\times\mathbf{Z}_2=\mathbf{Z}_2^3, & \left(\boldsymbol{\alpha}_2,\alpha_3\right)\in\mathbf{Z}_{2^2}\times\mathbf{Z}_2, & \vec{\mathbf{t}}_3=(\vec{\mathbf{t}}_2,t_3)\mathbf{Z}_2^2\times\mathbf{Z}_2=\mathbf{Z}_2^3, & (\mathbf{t}_2,t_3)\in\mathbf{Z}_{2^2}\times\mathbf{Z}_2,\\
\dots\dots & \dots\dots & \dots\dots & \dots\dots\\
\vec{\boldsymbol{\alpha}}_n=\left(\vec{\boldsymbol{\alpha}}_{n-1},\alpha_n\right)\in\mathbf{Z}_2^{n-1}\times\mathbf{Z}_2=\mathbf{Z}_2^n, & \left(\boldsymbol{\alpha}_{n-1},\alpha_n\right)\in\mathbf{Z}_{2^{n-1}}^1\times\mathbf{Z}_2, & \vec{\mathbf{t}}_n=(\vec{\mathbf{t}}_{n-1},t_n)\in\mathbf{Z}_2^{n-1}\times\mathbf{Z}_2=\mathbf{Z}_2^n, & (\mathbf{t}_{n-1},t_n)\in\mathbf{Z}_{2^{n-1}}^1\times\mathbf{Z}_2,
\end{array}$$

where $\mathbf{Z}_2^k=\{0,1\}^k=\mathbf{Z}_2^k$ and $\mathbf{Z}_{2^k}=\left\{0,1,2,...,2^k-1\right\}$.

Let $\left\{\mathrm{com}^{[n+1]}_{(\vec{\alpha}_n,0)}(\mathbf{t}_{n+1}),\ \mathrm{com}^{[n+1]}_{(\vec{\alpha}_n,1)}(\mathbf{t}_{n+1})\right\}_{\boldsymbol{\alpha}_n=0}^{2^n-1}$ be a set of 2^n pairs of complementary sequences of length 2^{n+1}.

Then the following matrix of depth $n+1$ has size $2^{n+1}\times 2^{n+1}$

$$\mathbf{G}^{[n+1]}_{2^{n+1}}=\boxplus_{\boldsymbol{\alpha}_{n+1}=0}^{2^{n+1}-1}\mathrm{com}^{[n+1]}_{\boldsymbol{\alpha}_{n+1}}(\mathbf{t}_{n+1})=\boxplus_{\boldsymbol{\alpha}_n=\mathbf{0}}^{2^n-1}\boxplus_{\alpha_{n+1}=0}^{1}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,\alpha_{n+1})}(\mathbf{t}_{n+1})=\boxplus_{\boldsymbol{\alpha}_n=\mathbf{0}}^{2^n-1}\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1})\end{bmatrix}=$$

$$=\boxplus_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1}\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,1)}(\mathbf{t}_{n+1})\\ \hline \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1})\end{bmatrix}=\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\mathbf{0},0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\mathbf{0},1)}(\mathbf{t}_{n+1})\\ \hline \mathrm{com}^{[n+1]}_{(\mathbf{1},0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\mathbf{1},1)}(\mathbf{t}_{n+1})\\ \hline \cdots\cdots\\ \cdots\cdots\\ \hline \mathrm{com}^{[n+1]}_{(\mathbf{2^n-1},0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\mathbf{2^n-1},1)}(\mathbf{t}_{n+1})\end{bmatrix}=\begin{bmatrix}\mathrm{com}^{[n+1]}_{(0,0,\ldots,0,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(0,0,\ldots,0,1)}(\mathbf{t}_{n+1})\\ \hline \mathrm{com}^{[n+1]}_{(0,0,\ldots,1,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(0,0,\ldots,1,1)}(\mathbf{t}_{n+1})\\ \hline \cdots\cdots\\ \cdots\cdots\\ \hline \mathrm{com}^{[n+1]}_{(1,1,\ldots,1,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(1,1,\ldots,1,1)}(\mathbf{t}_{n+1})\end{bmatrix} \quad (1)$$

and it is called the Golay matrix, where $\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1})\end{bmatrix}$ are a pair of complementary sequences and is the symbol of the vertical concatenation of $\left(2\times 2^{n+1}\right)-$ matrices $\begin{bmatrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1})\\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1})\end{bmatrix}$. For example,

$$\mathbf{G}^{[1]}_{2^1}=\begin{bmatrix}\mathrm{com}^{[1]}_0(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_1(\mathbf{t}_1)\end{bmatrix}=\boxplus_{\boldsymbol{\alpha}_1=0}^{1}\mathrm{com}^{[1]}_{\boldsymbol{\alpha}_1}(\mathbf{t}_1),\quad \mathbf{G}^{[2]}_{2^2}=\left[\mathrm{com}^{[2]}_{\boldsymbol{\alpha}_2}(\mathbf{t}_2)\right]^{3}_{\boldsymbol{\alpha}_2,\mathbf{t}_2=\mathbf{0}}=\boxplus_{\boldsymbol{\alpha}_1=\mathbf{0}}^{1}\begin{bmatrix}\mathrm{com}^{[2]}_{(\boldsymbol{\alpha}_1,0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(\boldsymbol{\alpha}_1,0)}(\mathbf{t}_2)\end{bmatrix}=\begin{bmatrix}\mathrm{com}^{[2]}_{(\mathbf{0},0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(\mathbf{0},1)}(\mathbf{t}_2)\\ \hline \mathrm{com}^{[2]}_{(\mathbf{1},0)}(\mathbf{t}_2)\\ \mathrm{com}^{[2]}_{(\mathbf{1},1)}(\mathbf{t}_2)\end{bmatrix},$$

$$\mathbf{G}^{[3]}_{2^3}=\left[\mathrm{com}^{[3]}_{\boldsymbol{\alpha}_3}(\mathbf{t}_3)\right]^{7}_{\boldsymbol{\alpha}_3,\mathbf{t}_3=\mathbf{0}}=\boxplus_{\boldsymbol{\alpha}_3=\mathbf{0}}^{7}\mathrm{com}^{[3]}_{\boldsymbol{\alpha}_3}(\mathbf{t}_3)=\boxplus_{\boldsymbol{\alpha}_2=\mathbf{0}}^{3}\begin{bmatrix}\mathrm{com}^{[3]}_{(\boldsymbol{\alpha}_2,0)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(\boldsymbol{\alpha}_2,1)}(\mathbf{t}_3)\end{bmatrix}=\begin{bmatrix}\mathrm{com}^{[3]}_{(\mathbf{0},0)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(\mathbf{0},1)}(\mathbf{t}_3)\\ \hline \mathrm{com}_{(\mathbf{1},0)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(\mathbf{1},1)}(\mathbf{t}_3)\\ \hline \mathrm{com}^{[3]}_{(\mathbf{2},0)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(\mathbf{2},1)}(\mathbf{t}_3)\\ \hline \mathrm{com}^{[3]}_{(\mathbf{3},0)}(\mathbf{t}_3)\\ \mathrm{com}^{[3]}_{(\mathbf{3},1)}(\mathbf{t}_3)\end{bmatrix}.$$

3. Method. Iteration construction for original Golay sequences

The matrix $\mathbf{G}^{[n+1]}_{2^{n+1}}$ is constructed by the following iteration construction:

$$\mathbf{G}^{[1]}_{2^1}\xrightarrow{\mathcal{F}_2}\mathbf{G}^{[2]}_{2^2}\xrightarrow{\mathcal{F}_2}\ldots\xrightarrow{\mathcal{F}_2}\mathbf{G}^{[n]}_{2^n}\xrightarrow{\mathcal{F}_2}\mathbf{G}^{[n+1]}_{2^{n+1}}. \quad (2)$$

The initial matrix $\mathbf{G}^{[1]}_{2^1}$ is formed by starting with the Fourier-Walsh $\left(2\times 2\right)-$matrix $\mathbf{G}^{[1]}_{2^1}=\mathcal{F}_2=$ $=\begin{bmatrix}\mathrm{com}^{[1]}_0(\mathbf{t}_1)\\ \mathrm{com}^{[1]}_1(\mathbf{t}_1)\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$ and by the repeated application of the iteration construction to pairs of rows in the matrix. Let us suppose that we have the Golay matrix $\mathbf{G}^{[n]}_{2^n}$. We need to construct the next Golay matrix $\mathbf{G}^{[n+1]}_{2^{n+1}}$ using only $\mathbf{G}^{[n]}_{2^n}$ and $\mathcal{F}_2=\mathbf{G}^{[1]}_{2^1}$. The Golay matrix $\mathbf{G}^{[n]}_{2^n}$ has a structure similar to (1):

$$\mathbf{G}_{2^n}^{[n]} = \coprod_{\boldsymbol{\alpha}_n=0}^{2^n-1} \mathrm{com}_{\boldsymbol{\alpha}_n}^{[n]}(\mathbf{t}_n) = \coprod_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_{n+1}) \end{bmatrix}. \tag{3}$$

To construct $\mathbf{G}_{2^{n+1}}^{[n+1]}$ from $\mathbf{G}_{2^n}^{[n]}$ we take each complementary pair of (3) in the form of $\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix}$ and construct shifted versa of their components

$$\begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n (0 \oplus_2 k)} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathbf{T}_{\mathbf{t}_n}^{2^n (1 \oplus_2 k)} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \mathrm{diag}\left\{ \mathbf{T}_{\mathbf{t}_n}^{2^n (0 \oplus_2 k)}, \mathbf{T}_{\mathbf{t}_n}^{2^n (1 \oplus_2 k)} \right\} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}\left(\mathbf{t}_n + 2^n (0 \oplus_2 \mathrm{k})\right) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}\left(\mathbf{t}_n + 2^n (1 \oplus_2 \mathrm{k})\right) \end{bmatrix},$$

where $k = 0,1$ and $\mathbf{T}_{\mathbf{t}_n}^{2^n s}$ is the shift operator on $2^n s$ positions in the time domain: $\mathbf{T}_{\mathbf{t}_n}^{2^n s} f(\mathbf{t}_n) := f(\mathbf{t}_n + 2^n \mathrm{s})$.

Now we construct the general building blocks for the Golay $\left(2^{n+1} \times 2^{n+1}\right)$-matrix $\mathbf{G}_{2^{n+1}}^{[n]}$:

$$\mathcal{F}_2 \cdot \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n (0 \oplus_2 k)} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathbf{T}_{\mathbf{t}_n}^{2^n (1 \oplus_2 k)} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \left[\mathrm{diag}\left\{ \mathbf{T}_{\mathbf{t}_n}^{2^n (0 \oplus_2 k)}, \mathbf{T}_{\mathbf{t}_n}^{2^n (1 \oplus_2 k)} \right\} \right] \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} =$$

$$= \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n (0 \oplus_2 k)} & \mathbf{T}_{\mathbf{t}_n}^{2^n (1 \oplus_2 k)} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n (0 \oplus_2 k)} & -\mathbf{T}_{\mathbf{t}_n}^{2^n (1 \oplus_2 k)} \end{bmatrix} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} = {}^{(k)}\mathcal{F}_2 \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix},$$

where

$${}^{(k)}\mathcal{F} = \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n (0 \oplus_2 k)} & \mathbf{T}_{\mathbf{t}_n}^{2^n (1 \oplus_2 k)} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n (0 \oplus_2 k)} & -\mathbf{T}_{\mathbf{t}_n}^{2^n (1 \oplus_2 k)} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \left[\mathrm{diag}\left\{ \mathbf{T}_{\mathbf{t}_n}^{2^n (0 \oplus_2 k)}, \mathbf{T}_{\mathbf{t}_n}^{2^n (1 \oplus_2 k)} \right\} \right].$$

Using building blocks of $\left(2^n \times 2^n\right)$-matrix $\mathbf{G}_{2^n}^{[n]}$, we construct the Golay $\left(2^{n+1} \times 2^{n+1}\right)$-matrix $\mathbf{G}_{2^{n+1}}^{[n+1]}$ according to the following iteration rule [16]:

$$\begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{bmatrix} {}^{(0)}\mathcal{F} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \\ \hline {}^{(1)}\mathcal{F} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \mathbf{T}_{\mathbf{t}_n}^{2^n} \\ \mathbf{I}_{\mathbf{t}_n} & -\mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n} & \mathbf{I}_{\mathbf{t}_n} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n} & -\mathbf{I}_{\mathbf{t}_n} \end{bmatrix} \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) \end{bmatrix} \end{bmatrix} =$$

$$= \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) - \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n) \\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n) \\ -\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n) \end{bmatrix} = \begin{bmatrix} \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}) \\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}) \end{bmatrix}, \tag{4}$$

where

$$\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}) = \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) + \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n),$$

$$\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}) = \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n) - \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n + 2^n),$$

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1}) = \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) + \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n),$$

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1}) = -\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) + \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n)$$

are complementary sequences of twice length, belonging to $\mathbf{G}^{[n+1]}_{2^{n+1}}$. Hence,

$$\mathbf{G}^{[n+1]}_{2^{n+1}} = \mathop{\boxplus}_{\boldsymbol{\alpha}_{n+1}=\mathbf{0}}^{\mathbf{2^{n+1}-1}} \left[\mathrm{com}^{[n+1]}_{\boldsymbol{\alpha}_{n+1}}(\mathbf{t}_{n+1})\right] = \mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{2^n-1} \begin{bmatrix} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1}) \end{bmatrix} = \mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1} \left[\begin{array}{c} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,1)}(\mathbf{t}_{n+1}) \\ \hline \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1}) \end{array}\right] =$$

$$= \mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{\mathbf{2^{n-1}-1}} \left[\begin{array}{c} \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} & \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} & -\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} \end{bmatrix} \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} & \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} & -\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} \end{bmatrix} \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \end{array}\right] = \mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1} \left[\begin{array}{c} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) + \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n + 2^n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) - \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n + 2^n) \\ \hline \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) + \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n) \\ -\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) + \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n) \end{array}\right] =$$

$$= \mathop{\boxplus}_{\alpha_{n-1}=0}^{2^{n-1}-1} \left(\mathop{\boxplus}_{\alpha_n=0}^{1} \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(0 \oplus_2 \alpha_n)} \cdot \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) + \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(1 \oplus_2 \alpha_n)} \cdot \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \\ \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(0 \oplus_2 \alpha_n)} \cdot \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) - \mathbf{T}_{\mathbf{t}_n}^{2^n\cdot(1 \oplus_2 \alpha_n)} \cdot \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \right). \tag{5}$$

This implies that

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_{n+1}) = \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n, t_{n+1}) = \sum_{t_{n+1}=0}^{1} (-1)^{\left(\alpha_n \oplus_2 t_{n+1}\right)\alpha_{n+1}} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \oplus_2 t_{n+1})}\left(\mathbf{t}_n + 2^n \cdot t_{n+1}\right). \tag{6}$$

Hence,

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n, t_{n+1}) = (-1)^{\left(\alpha_n \oplus_2 t_{n+1}\right)\alpha_{n+1}} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \oplus_2 t_{n+1})}\left(\mathbf{t}_n\right) = (-1)^{\alpha_n\alpha_{n+1}}(-1)^{\alpha_{n+1}t_{n+1}} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n \oplus_2 t_{n+1})}\left(\mathbf{t}_n\right). \tag{7}$$

It is finally a recurrent relation between complementary sequences of $\mathbf{G}^{[n+1]}_{2^{n+1}}$ and $\mathbf{G}^{[n]}_{2^n}$.

Remark 1. Obviously,

$$\begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \begin{matrix} \nearrow \\ \searrow \end{matrix} \left[\begin{array}{c} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,1)}(\mathbf{t}_{n+1}) \\ \hline \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1}) \end{array}\right] = \left[\begin{array}{c} {}^{(0)}\mathcal{F} \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \\ \hline {}^{(1)}\mathcal{F} \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \end{array}\right] = \left[\begin{array}{c} \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \mathbf{T}_{\mathbf{t}_n}^{2^n} \\ \mathbf{I}_{\mathbf{t}_n} & -\mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \\ \hline \begin{bmatrix} \mathbf{T}_{\mathbf{t}_n}^{2^n} & \mathbf{I}_{\mathbf{t}_n} \\ \mathbf{T}_{\mathbf{t}_n}^{2^n} & -\mathbf{I}_{\mathbf{t}_n} \end{bmatrix} \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \end{array}\right] =$$

$$= \left[\begin{array}{c} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \left(\mathbf{P}_2^0 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^0 \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \\ \hline \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \left(\mathbf{P}_2^1 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^1 \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \end{array}\right] = \left[\begin{array}{c} \mathcal{F}_2 \cdot \left(\mathbf{P}_2^0 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^0 \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \\ \hline \mathcal{F}_2 \cdot \left(\mathbf{P}_2^1 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^1 \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \end{array}\right],$$

where $\mathbf{P}_2^0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, $\mathbf{P}_2^1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ are 2-cyclic shift operators.

Hence,

$$\begin{bmatrix} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1}) \end{bmatrix} = \left[\mathcal{F}_2 \cdot \left(\mathbf{P}_2^{\alpha_n} \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}^{2^n}_{\mathbf{t}_n} \end{bmatrix} \cdot \mathbf{P}_2^{\alpha_n} \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \right] \tag{8}$$

and

$$\mathbf{G}^{[n+1]}_{2^{n+1}} = \mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{2^n-1} \begin{bmatrix} \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1}) \end{bmatrix} = \mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{2^n-1} \left[\mathcal{F}_2 \cdot \left(\mathbf{P}_2^{\alpha_n} \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}^{2^n}_{\mathbf{t}_n} \end{bmatrix} \cdot \mathbf{P}_2^{\alpha_n} \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n) \end{bmatrix} \right].$$

From (7) we obtain two expressions for $\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n)}(\mathbf{t}_n)$:

$$\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n)}(\mathbf{t}_n) = (-1)^{\sum_{i=1}^{n}{}^{\oplus} (\alpha_{i-1} \underset{2}{\oplus} t_i)(\alpha_i \underset{2}{\oplus} t_{i+1})}, \tag{9}$$

$$\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n)}(\mathbf{t}_n) = (-1)^{\sum_{i=1}^{n-1}{}^{\oplus} \alpha_i\alpha_{i+1}} \cdot (-1)^{\alpha_1 t_1 \underset{2}{\oplus} \alpha_2 t_2 \underset{2}{\oplus} \sum_{i=3}^{n}{}^{\oplus} (\alpha_i \underset{2}{\oplus} \alpha_{i-2}) t_i} \cdot (-1)^{\sum_{i=1}^{n-1}{}^{\oplus} t_i t_{i+1}} =$$

$$= (-1)^{\sum_{i=1}^{n-1}{}^{\oplus} \alpha_i\alpha_{i+1}} \cdot (-1)^{\langle \boldsymbol{\alpha} | R | \mathbf{t} \rangle} \cdot (-1)^{\sum_{i=1}^{n-1}{}^{\oplus} t_i t_{i+1}} = (-1)^{\sum_{i=1}^{n-1}{}^{\oplus} \alpha_i\alpha_{i+1}} \cdot (-1)^{\langle \boldsymbol{\beta} | \mathbf{t} \rangle} \cdot (-1)^{\sum_{i=1}^{n-1}{}^{\oplus} t_i t_{i+1}}, \tag{10}$$

where $\langle \boldsymbol{\alpha} | R | \mathbf{t} \rangle = \alpha_1 t_1 \underset{2}{\oplus} \alpha_2 t_2 \underset{2}{\oplus} \sum_{i=3}^{n} (\alpha_i \underset{2}{\oplus} \alpha_{i-2}) t_i = \sum_{i=1}^{n} \beta_i t_i$. Here $\vec{\boldsymbol{\beta}} = \vec{\boldsymbol{\alpha}} R$, where $R = \left[\delta_{i,j} \oplus \delta_{i,j+2} \right]_{i,j=1}^{n}$ and $\alpha_0, t_{n+1} \equiv 0$.

Example 1. It is easy to construct $\mathbf{G}^{[1]}_{2^1}$, $\mathbf{G}^{[2]}_{2^2}$ and $\mathbf{G}^{[3]}_{2^3}$:

$$\mathbf{G}^{[1]}_{2^1} \equiv \mathcal{F}_2 = \begin{bmatrix} \mathrm{com}^{[1]}_{(0)}(\mathbf{t}_1) \\ \mathrm{com}^{[1]}_{(1)}(\mathbf{t}_1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \left[\mathrm{com}^{[1]}_{(\alpha_1)}(t_1) \right] = \left[(-1)^{\alpha_1 t_1} \right],$$

$$\mathbf{G}^{[2]}_{2^2} = \begin{bmatrix} \mathrm{com}^{[2]}_{(0,0)}(\mathbf{t}_2) \\ \mathrm{com}^{[2]}_{(0,1)}(\mathbf{t}_2) \\ \mathrm{com}^{[2]}_{(1,0)}(\mathbf{t}_2) \\ \mathrm{com}^{[2]}_{(1,1)}(\mathbf{t}_2) \end{bmatrix} = \left[\begin{array}{cccc} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ \hdashline 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{array}\right] = \left[\mathrm{com}^{[2]}_{(\alpha_1,\alpha_2)}(t_1,t_2) \right] = \left[(-1)^{(\tilde{\alpha}_0 \underset{2}{\oplus} t_1)(\alpha_1 \underset{2}{\oplus} t_2)} (-1)^{(\alpha_1 \underset{2}{\oplus} t_2)(\alpha_2 \underset{2}{\oplus} \tilde{t}_3)} \right] =$$

$$= \mathbf{diag}\left\{ (-1)^{\alpha_1\alpha_2} \right\} \left[(-1)^{(\alpha_1 t_1 \underset{2}{\oplus} \alpha_2 t_2)} \right] \mathbf{diag}\left\{ (-1)^{t_1 t_2} \right\},$$

where $\tilde{\alpha}_0, \tilde{t}_3 \equiv 0$;

$$\mathbf{G}^{[3]}_{2^3} = \begin{bmatrix} \mathrm{com}^{[3]}_{(0,0,0)}(\mathbf{t}_3) \\ \mathrm{com}^{[3]}_{(0,0,1)}(\mathbf{t}_3) \\ \mathrm{com}^{[3]}_{(0,1,0)}(\mathbf{t}_3) \\ \mathrm{com}^{[3]}_{(0,1,1)}(\mathbf{t}_3) \\ \mathrm{com}^{[3]}_{(1,0,0)}(\mathbf{t}_3) \\ \mathrm{com}^{[3]}_{(1,0,1)}(\mathbf{t}_3) \\ \mathrm{com}^{[3]}_{(1,1,0)}(\mathbf{t}_3) \\ \mathrm{com}^{[3]}_{(1,1,1)}(\mathbf{t}_3) \end{bmatrix} = \left[\begin{array}{cccccccc} 1 & 1 & 1 & -1 & 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 \\ \hdashline 1 & 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ \hdashline 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & -1 & -1 & -1 \\ \hdashline -1 & 1 & 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 \end{array}\right] =$$

$$= \left[\mathrm{com}^{[3]}_{(\alpha_1,\alpha_2,\alpha_3)}(t_1,t_2,t_3) \right] = \left[(-1)^{(\alpha_0 \underset{2}{\oplus} t_1)(\alpha_1 \underset{2}{\oplus} t_2)} (-1)^{(\alpha_1 \underset{2}{\oplus} t_2)(\alpha_2 \underset{2}{\oplus} t_3)} (-1)^{(\alpha_2 \underset{2}{\oplus} t_3)(\alpha_3 \underset{2}{\oplus} t_4)} \right] =$$

$$= \mathbf{diag}\left\{ (-1)^{(\alpha_1\alpha_2 \underset{2}{\oplus} \alpha_2\alpha_3)} \right\} \cdot \left[(-1)^{\left(\alpha_1 t_1 \underset{2}{\oplus} \alpha_2 t_2 \underset{2}{\oplus} (\alpha_3 \underset{2}{\oplus} \alpha_1) t_3\right)} \right] \cdot \mathbf{diag}\left\{ (-1)^{(t_1 t_2 \underset{2}{\oplus} t_2 t_3)} \right\},$$

where $\tilde{\alpha}_0, \tilde{t}_4 \equiv 0$.

4. Generalization. Multi-parameter Golay sequences

In this section, we introduce the generalized Golay–Rudin–Shapiro sequences. They are represented by the following iteration construction

$$\mathbf{G}_{2^1}^{[1]}\left[\mathcal{CS}_2(\varphi_1,\alpha_1,\gamma_1)\right]\xrightarrow{\mathcal{CS}_2(\varphi_2,\alpha_2,\gamma_2)}\mathbf{G}_{2^2}^{[2]}\left[\mathcal{CS}_2(\varphi_1,\alpha_1,\gamma_1),\mathcal{CS}_2(\varphi_2,\alpha_2,\gamma_2)\right]\xrightarrow{\mathcal{CS}_2(\varphi_3,\alpha_3,\gamma_3)}\ldots$$

$$\xrightarrow{\mathcal{CS}_2(\varphi_n,\alpha_n,\gamma_n)}\mathbf{G}_{2^n}^{[n]}\left[\mathcal{CS}_2(\varphi_1,\alpha_1,\gamma_1),\ldots,\mathcal{CS}_2(\varphi_n,\alpha_n,\gamma_n)\right]\xrightarrow{\mathcal{CS}_2(\varphi_{n+1},\alpha_{n+1},\gamma_{n+1})}\mathbf{G}_{2^n}^{[n+1]}\left[\mathcal{CS}_2(\varphi_1,\alpha_1,\gamma_1),\ldots,\mathcal{CS}_2(\varphi_n,\alpha_n,\gamma_n),\mathcal{CS}_2(\varphi_{n+1},\alpha,\gamma_{n+1})\right],$$

based on a sequence of unitary transforms:

$$\begin{aligned}\mathcal{CS}_2(\varphi_k,\alpha_k,\gamma_k)&=\left[(-1)^{\alpha t}\cdot \mathrm{CS}_{\alpha,t}^{[k]}(\varphi_k,\alpha_k,\gamma_k)\right]_{\alpha,t=0}^{1}=\left[(-1)^{\alpha t}\cdot \mathrm{CS}_{\alpha,t}^{[k]}\right]_{\alpha,t=0}^{1}=\begin{bmatrix}\mathrm{C}_k & \mathrm{S}_k\\ \overline{\mathrm{S}}_k & -\overline{\mathrm{C}}_k\end{bmatrix}=\\ &=\begin{bmatrix}\mathrm{C}(\varphi_k,\alpha_k,\gamma_k) & \mathrm{S}(\varphi_k,\alpha_k,\gamma_k)\\ \overline{\mathrm{S}}(\varphi_k,\alpha_k,\gamma_k) & -\overline{\mathrm{C}}(\varphi_k,\alpha_k,\gamma_k)\end{bmatrix}=\begin{bmatrix}e^{i\alpha_k}\cos\varphi_k & e^{i\gamma_k}\sin\varphi_k\\ e^{-i\gamma_k}\sin\varphi_k & -e^{-i\alpha_k}\cos\varphi_k\end{bmatrix},\end{aligned}\qquad(11)$$

$\forall k=1,2,\ldots,n+1,$ where

$\mathrm{C}_k=\mathrm{C}(\varphi_k,\alpha_k,\gamma_k)=e^{i\alpha_k}\cos\varphi_k$, if $\alpha=0,\ t=0$; $\overline{\mathrm{C}}_k=\overline{\mathrm{C}}(\varphi_k,\alpha_k,\gamma_k)=e^{-i\alpha_k}\cos\varphi_k$, if $\alpha=0,\ t=1$;

$\mathrm{S}_k=\mathrm{S}(\varphi_k,\alpha_k,\gamma_k)=e^{i\gamma_k}\sin\varphi_k$, if $\alpha=1,\ t=0$; $\overline{\mathrm{S}}_k=\overline{\mathrm{S}}(\varphi_k,\alpha_k,\gamma_k)=e^{-i\gamma_k}\sin\varphi_k$, if $\alpha=1,\ t=1$.

For brevity, let $\theta_k:=(\varphi_k,\alpha_k,\gamma_k)$, $\boldsymbol{\theta}_k:=(\boldsymbol{\varphi}_k,\boldsymbol{\alpha}_k,\boldsymbol{\gamma}_k):=(\varphi_1,\alpha_1,\gamma_1;\ \varphi_2,\alpha_2,\gamma_2;\ \ldots;\varphi_k,\alpha_k,\gamma_k)$. As in the previous case we assume that we have the Golay matrix $\mathbf{G}_{2^n}^{[n]}[\boldsymbol{\theta}_n]=\mathbf{G}_{2^n}^{[n]}[\boldsymbol{\varphi}_n,\boldsymbol{\alpha}_n,\boldsymbol{\gamma}_n]$. We need to construct the next Golay matrix $\mathbf{G}_{2^n}^{[n+1]}[\boldsymbol{\theta}_{n+1}]$ using only $\mathbf{G}_{2^n}^{[n]}[\boldsymbol{\theta}_n]$ and $\mathcal{CS}_2(\theta_{n+1})$. We use the following iteration construction

$$\mathbf{G}_{2^{n+1}}^{[n+1]}(\boldsymbol{\theta}_{n+1})=\mathbf{G}_{2^{n+1}}^{[n+1]}(\boldsymbol{\theta}_n,\theta_{n+1})=\mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{2^n-1}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_n,0)}^{[n+1]}(\mathbf{t}_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_n,1)}^{[n+1]}(\mathbf{t}_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1})\end{bmatrix}=$$

$$=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,0)}^{[n+1]}(\mathbf{t}_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0,1)}^{[n+1]}(\mathbf{t}_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1})\\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,0)}^{[n+1]}(\mathbf{t}_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1})\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1,1)}^{[n+1]}(\mathbf{t}_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1})\end{bmatrix}=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1}\begin{bmatrix}\begin{bmatrix}\mathrm{C}(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} & \mathrm{S}(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1}\\ \overline{\mathrm{S}}(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0} & -\overline{\mathrm{C}}(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1}\end{bmatrix}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\end{bmatrix}\\ \hline \begin{bmatrix}\mathrm{C}(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} & \mathrm{S}(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0}\\ \overline{\mathrm{S}}(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 1} & -\overline{\mathrm{C}}(\theta_{n+1})\mathbf{T}_{\mathbf{t}_n}^{2^n\cdot 0}\end{bmatrix}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\\ \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\end{bmatrix}\end{bmatrix}=\qquad(12)$$

$$=\mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{\mathbf{2^{n-1}-1}}\begin{bmatrix}\begin{bmatrix}\mathrm{C}_{n+1} & \mathrm{S}_{n+1}\\ \overline{\mathrm{S}}_{n+1} & -\overline{\mathrm{C}}_{n+1}\end{bmatrix}\begin{bmatrix}\mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) & \\ \hline & \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n+2^n\,|\,\boldsymbol{\theta}_n)\end{bmatrix}\\ \hline \begin{bmatrix}\mathrm{C}_{n+1} & \mathrm{S}_{n+1}\\ \overline{\mathrm{S}}_{n+1} & -\overline{\mathrm{C}}_{n+1}\end{bmatrix}\begin{bmatrix} & \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},0)}^{[n]}(\mathbf{t}_n+2^n\,|\,\boldsymbol{\theta}_n)\\ \hline \mathrm{com}_{(\boldsymbol{\alpha}_{n-1},1)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) & \end{bmatrix}\end{bmatrix}=$$

$$=\left[\begin{array}{cc|cc|c|cc|cc}\mathrm{C}_{n+1} & \mathrm{S}_{n+1} & & & & & & & \\ \overline{\mathrm{S}}_{n+1} & -\overline{\mathrm{C}}_{n+1} & & & & & & & \\ \hline & & \mathrm{C}_{n+1} & \mathrm{S}_{n+1} & & & & & \\ & & \overline{\mathrm{S}}_{n+1} & -\overline{\mathrm{C}}_{n+1} & & & & & \\ \hline & & & & \ddots & & & & \\ \hline & & & & & \mathrm{C}_{n+1} & \mathrm{S}_{n+1} & & \\ & & & & & \overline{\mathrm{S}}_{n+1} & -\overline{\mathrm{C}}_{n+1} & & \\ \hline & & & & & & & \mathrm{C}_{n+1} & \mathrm{S}_{n+1}\\ & & & & & & & \overline{\mathrm{S}}_{n+1} & -\overline{\mathrm{C}}_{n+1}\end{array}\right]\cdot\left[\begin{array}{c|c}\mathrm{com}_{(\mathbf{0},0)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) & \\ \hline & \mathrm{com}_{(\mathbf{0},1)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\\ \hline & \mathrm{com}_{(\mathbf{0},0)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\\ \hline \mathrm{com}_{(\mathbf{0},1)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) & \\ \hline \vdots & \vdots\\ \hline \mathrm{com}_{(\mathbf{M},0)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) & \\ \hline & \mathrm{com}_{(\mathbf{M},1)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\\ \hline & \mathrm{com}_{(\mathbf{M},0)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\\ \hline \mathrm{com}_{(\mathbf{M},1)}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) & \end{array}\right],$$

where $\mathbf{M} = 2^{n-1}$. From (12) we obtain

$$\mathbf{G}^{[n+1]}_{2^{n+1}}(\boldsymbol{\theta}_n,\theta_{n+1}) = \mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1} \left[\frac{\begin{matrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,0)}(\mathbf{t}_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,1)}(\mathbf{t}_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1})\end{matrix}}{\begin{matrix}\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1}) \\ \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1})\end{matrix}}\right] = \tag{13}$$

$$= \mathop{\boxplus}_{\boldsymbol{\alpha}_{n-1}=0}^{2^{n-1}-1} \left[\frac{\begin{matrix}\mathrm{C}(\theta_{n+1})\cdot\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) + \mathrm{S}(\theta_{n+1})\cdot\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n + 2^n\,|\,\boldsymbol{\theta}_n) \\ \overline{\mathrm{S}}(\theta_{n+1})\cdot\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) - \overline{\mathrm{C}}(\theta_{n+1})\cdot\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n + 2^n\,|\,\boldsymbol{\theta}_n)\end{matrix}}{\begin{matrix}\mathrm{S}(\theta_{n+1})\cdot\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) + \mathrm{C}(\theta_{n+1})\cdot\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n\,|\,\boldsymbol{\theta}_n) \\ -\overline{\mathrm{C}}(\theta_{n+1})\cdot\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) + \overline{\mathrm{S}}(\theta_{n+1})\cdot\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n + 2^n\,|\,\boldsymbol{\theta}_n)\end{matrix}}\right].$$

Hence,

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n+1})}(\mathbf{t}_n, t_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1n+1}) = \mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n, t_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1}) =$$
$$= \sum_{t_{n+1}=0}^{1} (-1)^{(\alpha_n \oplus_2 t_{n+1})\alpha_{n+1}} \cdot \mathrm{CS}^{[n+1]}_{(\alpha_{n+1}\oplus_2\alpha_n),t_{n+1}}(\theta_{n+1})\cdot \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\oplus_2 t_{n+1})}\left(\mathbf{t}_n + 2^n\cdot t_{n+1}\,|\,\boldsymbol{\theta}_n\right)$$

and

$$\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n,\alpha_{n+1})}(\mathbf{t}_n, t_{n+1}\,|\,\boldsymbol{\theta}_n,\theta_{n+1}) = (-1)^{\left(\alpha_n \oplus_2 t_{n+1}\right)\alpha_{n+1}} \cdot \mathrm{CS}^{[n+1]}_{(\alpha_{n+1}\oplus_2\alpha_n),t_{n+1}}(\theta_{n+1})\cdot \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},\alpha_n\oplus_2 t_{n+1})}\left(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n\right). \tag{14}$$

This recurrent relation gives the following analytic expression for multi-parameter sequences

$$\mathrm{com}^{[n]}_{\boldsymbol{\alpha}_n}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n) = \prod_{i=1}^{n}(-1)^{(\alpha_{i-1}\oplus_2 t_i)(\alpha_i\oplus_2 t_{i+1})}\cdot \mathrm{CS}^{[n+1]}_{(\alpha_{i-1}\oplus_2\alpha_i),(t_i\oplus_2 t_{i+1})}(\theta_i). \tag{15}$$

Here are the particular cases:

1) if $\alpha_k = \gamma_k = 0,$ then

$$\mathcal{CS}_2(\theta_k) = \begin{bmatrix}\mathrm{C}_k(\theta_k) & \mathrm{S}_k(\theta_k)\\ \overline{\mathrm{S}}_k(\theta_k) & -\overline{\mathrm{C}}_k(\theta_k)\end{bmatrix} = \begin{bmatrix}\cos\varphi_k & \sin\varphi_k\\ \sin\varphi_k & -\cos\varphi_k\end{bmatrix} = \left[\mathrm{CS}_{\alpha,t}(\varphi_k)\right]^{1}_{\alpha,t=0},\quad \forall k = 1,2,...,n,$$

and

$$\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n)}(\mathbf{t}_n\,|\,\boldsymbol{\varphi}_n) = \prod_{i=1}^{n}(-1)^{\left(\alpha_{i-1}\oplus_2 t_i\right)\left(\alpha_i\oplus_2 t_{i+1}\right)}\cdot \mathrm{CS}^{[n+1]}_{(\alpha_{i-1}\oplus_2\alpha_i),(t_i\oplus_2 t_{i+1})}(\varphi_i); \tag{16}$$

2) if $\alpha_k = \gamma_k = 0,\ \varphi_k = \pi/4,$ then

$$\mathcal{CS}_2(\theta_k) = \begin{bmatrix}\mathrm{C}_k(\theta_k) & \mathrm{S}_k(\theta_k)\\ \overline{\mathrm{S}}_k(\theta_k) & -\overline{\mathrm{C}}_k(\theta_k)\end{bmatrix} = \begin{bmatrix}\cos(\pi/4) & \sin(\pi/4)\\ \sin(\pi/4) & -\cos(\pi/4)\end{bmatrix} = \frac{\sqrt{2}}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},\quad \forall k = 1,2,...,n,$$

and

$$\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n)}(\mathbf{t}_n) = \prod_{i=1}^{n}(-1)^{\left(\alpha_{i-1}\oplus_2 t_i\right)\left(\alpha_i\oplus_2 t_{i+1}\right)}. \tag{17}$$

Remark 2. For further generalization on m-complementary sequences we rewrite iteration rule(12) as

$$\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\end{bmatrix}\begin{matrix}\nearrow\\ \searrow\end{matrix}\left[\frac{\begin{bmatrix}\mathrm{C}(\theta_{n+1})\cdot\mathbf{I}_{\mathbf{t}_n} & \mathrm{S}(\theta_{n+1})\cdot\mathbf{T}^{2^n}_{\mathbf{t}_n}\\ \overline{\mathrm{S}}(\theta_{n+1})\cdot\mathbf{I}_{\mathbf{t}_n} & -\overline{\mathrm{C}}(\theta_{n+1})\cdot\mathbf{T}^{2^n}_{\mathbf{t}_n}\end{bmatrix}\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\end{bmatrix}}{\begin{bmatrix}\mathrm{C}(\theta_{n+1})\cdot\mathbf{T}^{2^n}_{\mathbf{t}_n} & \mathrm{S}(\theta_{n+1})\cdot\mathbf{I}_{\mathbf{t}_n}\\ \overline{\mathrm{S}}(\theta_{n+1})\cdot\mathbf{T}^{2^n}_{\mathbf{t}_n} & -\overline{\mathrm{C}}(\theta_{n+1})\cdot\mathbf{I}_{\mathbf{t}_n}\end{bmatrix}\begin{bmatrix}\mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n\,|\,\boldsymbol{\theta}_n)\end{bmatrix}}\right] =$$

$$
=\left[\begin{array}{c}
\begin{bmatrix} \mathrm{C}(\theta_{n+1}) & \mathrm{S}(\theta_{n+1}) \\ \bar{\mathrm{S}}(\theta_{n+1}) & -\bar{\mathrm{C}}(\theta_{n+1}) \end{bmatrix} \cdot \left(\mathbf{P}_2^0 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^0 \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \end{bmatrix} \\ \hline
\begin{bmatrix} \mathrm{C}(\theta_{n+1}) & \mathrm{S}(\theta_{n+1}) \\ \bar{\mathrm{S}}(\theta_{n+1}) & -\bar{\mathrm{C}}(\theta_{n+1}) \end{bmatrix} \cdot \left(\mathbf{P}_2^1 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^1 \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \end{bmatrix}
\end{array}\right] =
$$

$$
=\left[\begin{array}{c}
\mathcal{CS}_2(\theta_{n+1}) \cdot \left(\mathbf{P}_2^0 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^0 \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \end{bmatrix} \\ \hline
\mathcal{CS}_2(\theta_{n+1}) \cdot \left(\mathbf{P}_2^1 \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^1 \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \end{bmatrix}
\end{array}\right] \equiv \left[\begin{array}{c}
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,0)}(\mathbf{t}_{n+1} \mid \boldsymbol{\theta}_{n+1}) \\
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},0,1)}(\mathbf{t}_{n+1} \mid \boldsymbol{\theta}_{n+1}) \\ \hline
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,0)}(\mathbf{t}_{n+1} \mid \boldsymbol{\theta}_{n+1}) \\
\mathrm{com}^{[n+1]}_{(\boldsymbol{\alpha}_{n-1},1,1)}(\mathbf{t}_{n+1} \mid \boldsymbol{\theta}_{n+1})
\end{array}\right],
$$

where $\mathbf{P}_2^0 = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$, $\mathbf{P}_2^1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}$ are 2-cyclic shift operators. Hence,

$$
\begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1} \mid \boldsymbol{\theta}_{n+1}) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1} \mid \boldsymbol{\theta}_{n+1}) \end{bmatrix} = \left[\mathcal{CS}_2(\theta_{n+1}) \cdot \left(\mathbf{P}_2^{\alpha_n} \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^{\alpha_n} \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \end{bmatrix} \right] \tag{18}
$$

and

$$
\mathbf{G}^{[n+1]}_{2^{n+1}}(\boldsymbol{\theta}_{n+1}) = \mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{2^n\text{-}1} \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n,0)}(\mathbf{t}_{n+1} \mid \boldsymbol{\theta}_{n+1}) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_n,1)}(\mathbf{t}_{n+1} \mid \boldsymbol{\theta}_{n+1}) \end{bmatrix} = \mathop{\boxplus}_{\boldsymbol{\alpha}_n=0}^{2^n\text{-}1} \left[\mathcal{CS}_2(\theta_{n+1}) \cdot \left(\mathbf{P}_2^{\alpha_n} \cdot \begin{bmatrix} \mathbf{I}_{\mathbf{t}_n} & \\ & \mathbf{T}_{\mathbf{t}_n}^{2^n} \end{bmatrix} \cdot \mathbf{P}_2^{\alpha_n} \right) \cdot \begin{bmatrix} \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},0)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \\ \mathrm{com}^{[n]}_{(\boldsymbol{\alpha}_{n-1},1)}(\mathbf{t}_n \mid \boldsymbol{\theta}_n) \end{bmatrix} \right].
$$

Example 3. It is easy to construct $\mathbf{G}^{[1]}_{2^1}[\boldsymbol{\theta}_1], \mathbf{G}^{[2]}_{2^2}[\boldsymbol{\theta}_2], \mathbf{G}^{[3]}_{2^3}[\boldsymbol{\theta}_3]$:

$$
\mathbf{G}^{[1]}_{2^1}(\theta_1) = \left[\mathrm{com}^{[1]}_{(\alpha_1)}(\mathbf{t}_1 \mid \theta_1) \right] = \begin{bmatrix} \mathrm{com}_{(0)}(\mathbf{t}_1 \mid \theta_1) \\ \mathrm{com}_{(1)}(\mathbf{t}_1 \mid \theta_1) \end{bmatrix} = \begin{bmatrix} \mathrm{C}_1(\theta_1) & \mathrm{S}_1(\theta_1) \\ \bar{\mathrm{S}}_1(\theta_1) & -\bar{\mathrm{C}}_1(\theta_1) \end{bmatrix} = \begin{bmatrix} \mathrm{C}_k(\varphi_k,\alpha_k) & \mathrm{S}_k(\varphi_k,\gamma_k) \\ \bar{\mathrm{S}}_k(\varphi_k,\gamma_k) & -\bar{\mathrm{C}}_k(\varphi_k,\alpha_k) \end{bmatrix} =
$$

$$
= \begin{bmatrix} e^{i\alpha_k}\cos\varphi_k & e^{i\gamma_k}\sin\varphi_k \\ e^{-i\gamma_k}\sin\varphi_k & -e^{-i\alpha_k}\cos\varphi_k \end{bmatrix} = \left[(-1)^{\alpha t} \cdot \mathrm{CS}^{[1]}_{\alpha,t}(\theta_1) \right]^{1}_{\alpha,t=0};
$$

$$
\mathbf{G}^{[2]}_{2^2}(\theta_1,\theta_2) = \left[\mathrm{com}^{[2]}_{(\alpha_1,\alpha_2)}(\mathbf{t}_2 \mid \varphi_1,\alpha_1,\gamma_1;\varphi_2\alpha_2\gamma_2) \right] = \begin{bmatrix} \mathrm{com}_{(0,0)}(\mathbf{t}_2 \mid \theta_1,\theta_2) \\ \mathrm{com}_{(0,1)}(\mathbf{t}_2 \mid \theta_1,\theta_2) \\ \mathrm{com}_{(1,0)}(\mathbf{t}_2 \mid \theta_1,\theta_2) \\ \mathrm{com}_{(1,1)}(\mathbf{t}_2 \mid \theta_1,\theta_2) \end{bmatrix} =
$$

$$
= \left[(-1)^{(\alpha_1 \oplus_2 t_2)(\alpha_2 \oplus_2 \tilde{t}_3)} \cdot (-1)^{(\tilde{\alpha}_0 \oplus_2 t_1)(\alpha_1 \oplus_2 t_2)} \cdot \mathrm{CS}^{[2]}_{(\alpha_1 \oplus_2 \alpha_2),(t_2 \oplus_2 \tilde{t}_3)}(\theta_2) \cdot \mathrm{CS}^{[1]}_{(\tilde{\alpha}_0 \oplus_2 \alpha_1),(t_1 \oplus_2 t_2)}(\theta_1) \right] =
$$

$$
= \left[\begin{array}{cc|cc} \mathrm{C}_2 & \mathrm{S}_2 & & \\ \bar{\mathrm{S}}_2 & -\bar{\mathrm{C}}_2 & & \\ \hline & & \mathrm{C}_2 & \mathrm{S}_2 \\ & & \bar{\mathrm{S}}_2 & -\bar{\mathrm{C}}_2 \end{array}\right] \left[\begin{array}{cc|cc} \mathrm{C}_1 & \mathrm{S}_1 & & \\ & & \bar{\mathrm{S}}_1 & -\bar{\mathrm{C}}_1 \\ \hline & & \mathrm{C}_1 & \mathrm{S}_1 \\ \bar{\mathrm{S}}_1 & -\bar{\mathrm{C}}_1 & & \end{array}\right] = \left[\begin{array}{cc|cc} \mathrm{C}_2\mathrm{C}_1 & \mathrm{C}_2\mathrm{S}_1 & \mathrm{S}_2\bar{\mathrm{S}}_1 & -\mathrm{S}_2\bar{\mathrm{C}}_1 \\ \bar{\mathrm{S}}_2\mathrm{C}_1 & \bar{\mathrm{S}}_2\mathrm{S}_1 & -\bar{\mathrm{C}}_2\bar{\mathrm{S}}_1 & \bar{\mathrm{C}}_2\bar{\mathrm{C}}_1 \\ \hline \mathrm{S}_2\bar{\mathrm{S}}_1 & -\mathrm{S}_2\bar{\mathrm{C}}_1 & \mathrm{C}_2\mathrm{C}_1 & \mathrm{C}_2\mathrm{S}_1 \\ -\bar{\mathrm{C}}_2\bar{\mathrm{S}}_1 & \bar{\mathrm{C}}_2\bar{\mathrm{C}}_1 & \bar{\mathrm{S}}_2\mathrm{C}_1 & \bar{\mathrm{S}}_2\mathrm{S}_1 \end{array}\right] =
$$

$$
= \left[\begin{array}{cc|cc}
e^{i\alpha_2}e^{i\alpha_1}\cos\varphi_2\cos\varphi_1 & e^{i\alpha_2}e^{i\gamma_1}\cos\varphi_2\sin\varphi_1 & e^{i\gamma_2}e^{-i\gamma_1}\sin\varphi_2\sin\varphi_1 & -e^{i\gamma_2}e^{-i\alpha_1}\sin\varphi_2\cos\varphi_1 \\
e^{-i\gamma_2}e^{i\alpha_1}\sin\varphi_2\cos\varphi_1 & e^{-i\gamma_2}e^{i\gamma_1}\sin\varphi_2\sin\varphi_1 & -e^{-i\alpha_2}e^{-i\gamma_1}\cos\varphi_2\sin\varphi_1 & e^{-i\alpha_2}e^{-i\alpha_1}\cos\varphi_2\cos\varphi_1 \\ \hline
e^{i\gamma_2}e^{-i\gamma_1}\sin\varphi_2\sin\varphi_1 & -e^{i\gamma_2}e^{-i\alpha_1}\sin\varphi_2\cos\varphi_1 & e^{i\alpha_2}e^{i\alpha_1}\cos\varphi_2\cos\varphi_1 & e^{i\alpha_2}e^{i\gamma_1}\cos\varphi_2\sin\varphi_1 \\
-e^{-i\alpha_2}e^{-i\gamma_1}\cos\varphi_2\sin\varphi_1 & e^{-i\alpha_2}e^{-i\alpha_1}\cos\varphi_2\cos\varphi_1 & e^{-i\gamma_2}e^{i\alpha_1}\sin\varphi_2\cos\varphi_1 & e^{-i\gamma_2}e^{i\gamma_1}\sin\varphi_2\sin\varphi_1
\end{array}\right];
$$

where $\tilde{\alpha}_0, \tilde{t}_3 \equiv 0$;

$$\mathbf{G}_{2^3}^{[3]}(\theta_1,\theta_2,\theta_3)=\begin{bmatrix}\mathrm{com}_{(0,0,0)}(\mathbf{t}_3\,|\,\varphi_1,\alpha_1,\gamma_1;\varphi_2,\alpha_2,\gamma_2;\varphi_3,\alpha_3,\gamma_3)\\ \mathrm{com}_{(0,0,1)}(\mathbf{t}_3\,|\,\varphi_1,\alpha_1,\gamma_1;\varphi_2,\alpha_2,\gamma_2;\varphi_3,\alpha_3,\gamma_3)\\ \mathrm{com}_{(0,1,0)}(\mathbf{t}_3\,|\,\varphi_1,\alpha_1,\gamma_1;\varphi_2,\alpha_2,\gamma_2;\varphi_3,\alpha_3,\gamma_3)\\ \mathrm{com}_{(0,1,1)}(\mathbf{t}_3\,|\,\varphi_1,\alpha_1,\gamma_1;\varphi_2,\alpha_2,\gamma_2;\varphi_3,\alpha_3,\gamma_3)\\ \mathrm{com}_{(1,0,0)}(\mathbf{t}_3\,|\,\varphi_1,\alpha_1,\gamma_1;\varphi_2,\alpha_2,\gamma_2;\varphi_3,\alpha_3,\gamma_3)\\ \mathrm{com}_{(1,0,1)}(\mathbf{t}_3\,|\,\varphi_1,\alpha_1,\gamma_1;\varphi_2,\alpha_2,\gamma_2;\varphi_3,\alpha_3,\gamma_3)\\ \mathrm{com}_{(1,1,0)}(\mathbf{t}_3\,|\,\varphi_1,\alpha_1,\gamma_1;\varphi_2,\alpha_2,\gamma_2;\varphi_3,\alpha_3,\gamma_3)\\ \mathrm{com}_{(1,1,1)}(\mathbf{t}_3\,|\,\varphi_1,\alpha_1,\gamma_1;\varphi_2,\alpha_2,\gamma_2;\varphi_3,\alpha_3,\gamma_3)\end{bmatrix}=\left[\begin{array}{cc:cc:cc:cc} C_3 & S_3 & & & & & & \\ \bar{S}_3 & -\bar{C}_3 & & & & & & \\ \hdashline & & C_3 & S_3 & & & & \\ & & \bar{S}_3 & -\bar{C}_3 & & & & \\ \hdashline & & & & C_3 & S_3 & & \\ & & & & \bar{S}_3 & -\bar{C}_3 & & \\ \hdashline & & & & & & C_3 & S_3 \\ & & & & & & \bar{S}_3 & -\bar{C}_3 \end{array}\right]\cdot$$

$$\cdot\left[\begin{array}{cccc|cccc} C_2C_1 & C_2S_1 & S_2\bar{S}_1 & -S_2\bar{C}_1 & & & & \\ & & & & \bar{S}_2C_1 & \bar{S}_2S_1 & -\bar{C}_2\bar{S}_1 & \bar{C}_2\bar{C}_1 \\ \hdashline & & & & C_2C_1 & C_2S_1 & S_2\bar{S}_1 & -S_2\bar{C}_1 \\ \bar{S}_2C_1 & \bar{S}_2S_1 & -\bar{C}_2\bar{S}_1 & \bar{C}_2\bar{C}_1 & & & & \\ \hdashline S_2\bar{S}_1 & -S_2\bar{C}_1 & C_2C_1 & C_2S_1 & & & & \\ & & & & -\bar{C}_2\bar{S}_1 & \bar{C}_2\bar{C}_1 & \bar{S}_2C_1 & \bar{S}_2S_1 \\ \hdashline & & & & S_2\bar{S}_1 & -S_2\bar{C}_1 & C_2C_1 & C_2S_1 \\ -\bar{C}_2\bar{S}_1 & \bar{C}_2\bar{C}_1 & \bar{S}_2C_1 & \bar{S}_2S_1 & & & & \end{array}\right]=$$

$$=\left[\begin{array}{cccc|cccc} C_3C_2C_1 & C_3C_2S_1 & C_3S_2\bar{S}_1 & -C_3S_2\bar{C}_1 & S_3\bar{S}_2C_1 & S_3\bar{S}_2S_1 & -S_3\bar{C}_2\bar{S}_1 & S_3\bar{C}_2\bar{C}_1 \\ \bar{S}_3C_2C_1 & \bar{S}_3C_2S_1 & \bar{S}_3S_2\bar{S}_1 & -\bar{S}_3S_2\bar{C}_1 & -\bar{C}_3\bar{S}_2C_1 & -\bar{C}_3\bar{S}_2S_1 & \bar{C}_3\bar{C}_2\bar{S}_1 & -\bar{C}_3\bar{C}_2\bar{C}_1 \\ \hdashline S_3\bar{S}_2C_1 & S_3\bar{S}_2S_1 & -S_3\bar{C}_2\bar{S}_1 & S_3\bar{C}_2\bar{C}_1 & C_3C_2C_1 & C_3C_2S_1 & C_3S_2\bar{S}_1 & -C_3S_2\bar{C}_1 \\ -\bar{C}_3\bar{S}_2C_1 & -\bar{C}_3\bar{S}_2S_1 & \bar{C}_3\bar{C}_2\bar{S}_1 & -\bar{C}_3\bar{C}_2\bar{C}_1 & \bar{S}_3C_2C_1 & \bar{S}_3C_2S_1 & \bar{S}_3S_2\bar{S}_1 & -\bar{S}_3S_2\bar{C}_1 \\ \hdashline C_3S_2\bar{S}_1 & -C_3S_2\bar{C}_1 & C_3C_2C_1 & C_3C_2S_1 & -S_3\bar{C}_2\bar{S}_1 & S_3\bar{C}_2\bar{C}_1 & S_3\bar{S}_2C_1 & S_3\bar{S}_2S_1 \\ \bar{S}_3S_2\bar{S}_1 & -\bar{S}_3S_2\bar{C}_1 & \bar{S}_3C_2C_1 & \bar{S}_3C_2S_1 & \bar{C}_3\bar{C}_2\bar{S}_1 & -\bar{C}_3\bar{C}_2\bar{C}_1 & -\bar{C}_3\bar{S}_2C_1 & -\bar{C}_3\bar{S}_2S_1 \\ \hdashline -S_3\bar{C}_2\bar{S}_1 & S_3\bar{C}_2\bar{C}_1 & S_3\bar{S}_2C_1 & S_3\bar{S}_2S_1 & C_3S_2\bar{S}_1 & -C_3S_2\bar{C}_1 & C_3C_2C_1 & C_3C_2S_1 \\ \bar{C}_3\bar{C}_2\bar{S}_1 & -\bar{C}_3\bar{C}_2\bar{C}_1 & -\bar{C}_3\bar{S}_2C_1 & -\bar{C}_3\bar{S}_2S_1 & \bar{S}_3S_2\bar{S}_1 & -\bar{S}_3S_2\bar{C}_1 & \bar{S}_3C_2C_1 & \bar{S}_3C_2S_1 \end{array}\right].$$

Unitary transforms (11)

$$\mathcal{CS}_2(\varphi,\alpha,\gamma)=\left[(-1)^{\alpha t}\cdot\mathrm{CS}_{\alpha,t}\right]_{\alpha,t=0}^{1}=\begin{bmatrix}\mathrm{C} & \mathrm{S}\\ \bar{\mathrm{S}} & -\bar{\mathrm{C}}\end{bmatrix}=\begin{bmatrix}e^{i\alpha}\cos\varphi & e^{i\gamma}\sin\varphi\\ e^{-i\gamma}\sin\varphi & -e^{-i\alpha}\cos\varphi\end{bmatrix}$$

form the group $SU(2,\mathbf{C})$ of all (2×2)-unitary matrices with complex entries and determinant equal to $\pm$ one:

$$SU(2)=\left\{\mathcal{CS}_2\in Mat(2,\mathbf{C})\,|\,\left(\mathcal{CS}_2^\dagger\cdot\mathcal{CS}_2=\mathcal{CS}_2\cdot\mathcal{CS}_2^\dagger=\mathbf{I}_2\right)\&\left(\det(\mathcal{CS}_2)=\pm 1\right)\right\}.$$

Let us introduce (2×2)-unitary matrices with $\mathcal{A}lg$ -valued entries and determinant equal to $\pm$ one:

$$\mathcal{CS}_2(\sigma,{}^1\varepsilon,{}^2\varepsilon)=\begin{bmatrix}{}^1\varepsilon\cdot\sigma & {}^2\varepsilon\cdot\tilde{\sigma}\\ {}^2\bar{\varepsilon}\cdot\tilde{\sigma} & -{}^1\bar{\varepsilon}\cdot\sigma\end{bmatrix}=\begin{bmatrix}\mathrm{C}_k(\sigma,{}^1\varepsilon) & \mathrm{S}_k(\sigma,{}^2\varepsilon)\\ \bar{\mathrm{S}}_k(\sigma,{}^2\varepsilon) & -\bar{\mathrm{C}}_k(\sigma,{}^1\varepsilon)\end{bmatrix}=\left[(-1)^{\alpha t}\cdot\mathrm{CS}_{\alpha,t}(\sigma,{}^1\varepsilon,{}^2\varepsilon)\right]_{\alpha,t=0}^{1}=$$

$$=\left[(-1)^{\alpha t}\cdot\mathrm{CS}_{\alpha,t}\right]_{\alpha,t=0}^{1}=\begin{bmatrix}\mathrm{C} & \mathrm{S}\\ \bar{\mathrm{S}} & -\bar{\mathrm{C}}\end{bmatrix},$$

where ${}^1\varepsilon,{}^2\varepsilon\in\mathcal{A}lg,\ {}^1\varepsilon\cdot{}^1\bar{\varepsilon}=\left|{}^1\varepsilon\right|^2=1,\ {}^2\varepsilon\cdot{}^2\bar{\varepsilon}=\left|{}^2\varepsilon\right|^2=1,\ \sigma,\tilde{\sigma}\in\mathcal{A}lg,\ \sigma^2+\tilde{\sigma}^2=1,$ and $\mathcal{A}lg$ is an algebra (for example, Clifford algebras or finite Galois fields). In this case

$$\mathbf{G}_{2^n}^{[n]}(\boldsymbol{\sigma}_n,{}^1\boldsymbol{\varepsilon}_n,{}^2\boldsymbol{\varepsilon}_n)=\left[\mathrm{com}_{\boldsymbol{\alpha}_n}^{[n]}(\mathbf{t}_n\,|\,\boldsymbol{\sigma}_n,{}^1\boldsymbol{\varepsilon}_n,{}^2\boldsymbol{\varepsilon}_n)\right]=\left[\prod_{i=1}^{n}(-1)^{(\alpha_{i-1}\oplus_2 t_i)(\alpha_i\oplus_2 t_{i+1})}\cdot\mathrm{CS}^{[n+1]}_{(\alpha_{i-1}\oplus_2\alpha_i),(t_i\oplus_2 t_{i+1})}(\sigma_i,{}^1\varepsilon_i,{}^2\varepsilon_i)\right]$$

is the Fourier-Galois-Golay algebraic transform (FGGAT).

Example 4. Let $\mathcal{A}lg = \mathbf{GF}(7),\ \sigma = 2,\ \tilde{\sigma} = 5,\ {}^1\varepsilon, {}^2\varepsilon \equiv 1$. It is easy to check that $\sigma^2 + \tilde{\sigma}^2 = 2^2 + 5^2 = 1$ $(\bmod 7)$. If $\mathcal{CS}_2(\sigma_1, {}^1\varepsilon_1, {}^2\varepsilon_1) = \mathcal{CS}_2(\sigma_2, {}^1\varepsilon_2, {}^2\varepsilon_2) = \begin{bmatrix} 2 & 5 \\ 5 & -2 \end{bmatrix}$, then

$$\mathbf{G}_{2^2}^{[2]} = \left[\mathrm{com}_{(\alpha_1,\alpha_2)}^{[2]}(\mathbf{t}_2)\right] = \begin{bmatrix} \mathrm{com}_{(0,0)}(\mathbf{t}_2) \\ \mathrm{com}_{(0,1)}(\mathbf{t}_2) \\ \mathrm{com}_{(1,0)}(\mathbf{t}_2) \\ \mathrm{com}_{(1,1)}(\mathbf{t}_2) \end{bmatrix} = \left[\begin{array}{cc|cc} 2 & 5 & & \\ 5 & -2 & & \\ \hline & & 2 & 5 \\ & & 5 & -2 \end{array}\right] \left[\begin{array}{cc|cc} 2 & 5 & & \\ & & 5 & -2 \\ \hline & & 2 & 5 \\ 5 & -2 & & \end{array}\right] = \left[\begin{array}{cc|cc} 4 & 10 & 4 & -10 \\ 10 & 4 & -10 & 4 \\ \hline 4 & -10 & 4 & 10 \\ -10 & 4 & 10 & 4 \end{array}\right].$$

is the Fourier-Galois-Golay algebraic transform.

If $\mathcal{A}lg = Clif$, where $Clif$ is the Clifford algebra, then $\mathbf{G}_{2^n}^{[n]}(\boldsymbol{\sigma}_n, {}^1\boldsymbol{\varepsilon}_n, {}^2\boldsymbol{\varepsilon}_n)$ is the Fourier-Clifford-Golay transform (FCGT), if $\mathcal{A}lg = Ham$, where Ham is the quaternion Hamilton algebra, then $\mathbf{G}_{2^n}^{[n]}(\boldsymbol{\sigma}_n, {}^1\boldsymbol{\varepsilon}_n, {}^2\boldsymbol{\varepsilon}_n)$ is the Fourier-Hamilton-Golay transform (FHGT) and so on.

5. Conclusion

In this paper, we have shown a new unified approach to the so-called generalized complex-$\mathbf{GF}(p)$-or Clifford-valued complementary sequences. The approach is based on a new iteration generating construction. This construction has a rich algebraic structure. It is associated not with the triple $\left(\mathbf{Z}_2^n, \mathcal{F}_2, \mathbf{C}\right)$, but with

$$\left(\mathbf{Z}_2, \mathcal{CS}_2(\varphi,\alpha,\gamma), \mathcal{A}lg\right) \text{ and } \left(\mathbf{Z}_2, \left\{\mathcal{CS}_2^1(\varphi_1,\alpha_1,\gamma_1), \mathcal{CS}_2^2(\varphi_2,\alpha_2,\gamma_2), ..., \mathcal{CS}_n^1(\varphi_n,\alpha_n,\gamma_n)\right\}, \mathcal{A}lg\right),$$

where $\mathcal{CS}_2^1(\varphi_1,\alpha_1,\gamma_1)$ and $\left\{\mathcal{CS}_2^1(\varphi_1,\alpha_1,\gamma_1), \mathcal{CS}_2^2(\varphi_2,\alpha_2,\gamma_2), ..., \mathcal{CS}_n^1(\varphi_n,\alpha_n,\gamma_n)\right\}$ are a single transform or a set of arbitrary unitary (2×2)-transforms of type $\mathcal{CS}_2(\varphi,\alpha,\gamma) = \begin{bmatrix} e^{i\alpha}\cos\varphi & e^{i\gamma}\sin\varphi \\ e^{-i\gamma}\sin\varphi & -e^{-i\alpha}\cos\varphi \end{bmatrix}$, if $\mathcal{A}lg = \mathbf{C}$; or orthogonal (2×2)-transforms of type $\mathcal{CS}_2(\sigma, {}^1\varepsilon, {}^2\varepsilon) = \begin{bmatrix} {}^1\varepsilon\cdot\sigma & {}^2\varepsilon\cdot\tilde{\sigma} \\ {}^2\bar{\varepsilon}\cdot\tilde{\sigma} & -{}^1\bar{\varepsilon}\cdot\sigma \end{bmatrix}$, if $\mathcal{A}lg$ is a finite algebra (for example, a finite Galois field $\mathbf{GF}(q)$, or a finite Clifford algebra), where ${}^1\varepsilon, {}^2\varepsilon \in \mathcal{A}lg,\ {}^1\varepsilon\cdot{}^1\bar{\varepsilon} = \left|{}^1\varepsilon\right|^2 = 1,\ {}^2\varepsilon\cdot{}^2\bar{\varepsilon} = \left|{}^2\varepsilon\right|^2 = 1,\ \sigma,\tilde{\sigma} \in \mathcal{A}lg,\ \sigma^2 + \tilde{\sigma}^2 = 1$.

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